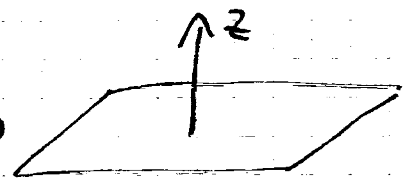
Midterm Solutions

1. a)

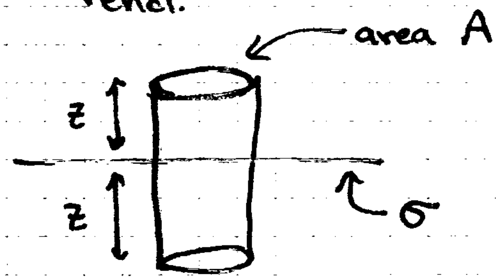


Symmetries

translation in x & y $\Rightarrow$ no dependence on x, y

rotation about z $\rightarrow$ no E_x or E_y components.

Gauss's Law $\quad \epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = Q_{encl.}$



Choose S as shown

Since $E = E_z \vec{a}_z$

$\vec{E} \cdot d\vec{S} = 0$ for the sidewall and

$\vec{E} \cdot d\vec{S} = |E_z| dS$ for the end caps $\quad E_z$ constant.

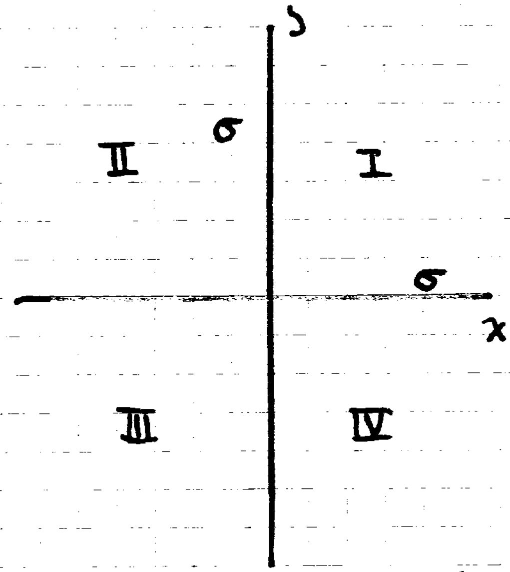
Also $|E_z|$ is the same for the top and bottom endcap because of symmetry.

so $\quad \epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = 2 A E_z \epsilon_0$

$Q_{encl.} = A\sigma$

so $\quad E_z = \dfrac{\sigma}{2\epsilon_0} \qquad \vec{E} = \pm \dfrac{\sigma}{2\epsilon_0} \vec{a}_z \quad$ + for $z > 0$, − for $z < 0$

b)



Use super position

The field from the sheet in the x-z plane is

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0}\vec{a}_y \qquad \begin{matrix} + & y>0 \\ - & y<0 \end{matrix}$$

The field from the sheet in the y-z plane is

$$\vec{E} = \pm \frac{\sigma}{2\epsilon_0}\vec{a}_x \qquad \begin{matrix} + & x>0 \\ - & x<0 \end{matrix}$$

so in

Region I $\quad \vec{E} = \frac{\sigma}{2\epsilon_0}(\vec{a}_x + \vec{a}_y)$

Region II $\quad \vec{E} = \frac{\sigma}{2\epsilon_0}(-\vec{a}_x + \vec{a}_y)$

Region III $\quad \vec{E} = \frac{\sigma}{2\epsilon_0}(-\vec{a}_x - \vec{a}_y)$

Region IV $\quad \vec{E} = \frac{\sigma}{2\epsilon_0}(\vec{a}_x - \vec{a}_y)$

c) Again use super position.

For the plane of charge $\quad \vec{E} = \frac{\pm\sigma}{2\epsilon_0}E_z \qquad \begin{matrix} + & z>0 \\ - & z<0 \end{matrix}$

For the sphere

$$\vec{E} = \frac{\frac{4}{3}\pi a^3 \rho}{4\pi\epsilon_0 r^2}\vec{a}_r = \frac{a^3 \rho}{3\epsilon_0 r^2}\vec{a}_r$$

Since a uniform sphere of charge produces the field of a point charge outside the sphere

Along the z-axis $\vec{a}_r$ & $\vec{a}_z$ point in the same direction (or opposite directions for $z<0$) so along the z-axis outside the sphere

$$\vec{E} = \pm\left(\frac{\sigma}{2\epsilon_0} + \frac{a^3\rho}{3\epsilon_0 z^2}\right)\vec{a}_z \qquad \begin{matrix} + & z>0 \\ - & z<0 \end{matrix}$$

2. $5nC$ @ $(0,1,0)$
$-5nC$ @ $(1,1,0)$

find $\vec{E}$ @ $(1,0,1)$

$$\vec{r} = \vec{a}_x + \vec{a}_z$$

$5nC$ $\quad \vec{r}\,' = \vec{a}_y \qquad \vec{r}-\vec{r}\,' = \vec{a}_x - \vec{a}_y + \vec{a}_z \qquad |\vec{r}-\vec{r}\,'| = \sqrt{3}$

so

$$\vec{E}_1 = \frac{5nC}{4\pi\epsilon_0}\,\frac{\vec{a}_x - \vec{a}_y + \vec{a}_z}{3\sqrt{3}}$$

$-5nC$ $\quad \vec{r}\,' = \vec{a}_x + \vec{a}_y \qquad \vec{r}-\vec{r}\,' = -\vec{a}_y + \vec{a}_z \qquad |\vec{r}-\vec{r}\,'| = \sqrt{2}$

$$\vec{E}_2 = \frac{-5nC}{4\pi\epsilon_0}\,\frac{-\vec{a}_y + \vec{a}_z}{2\sqrt{2}}$$

so

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 = \frac{5nC}{4\pi\epsilon_0}\left(\frac{1}{3\sqrt{3}}\vec{a}_x + \left(\frac{-1}{3\sqrt{3}} + \frac{1}{2\sqrt{2}}\right)\vec{a}_y + \left(\frac{1}{3\sqrt{3}} - \frac{1}{2\sqrt{2}}\right)\vec{a}_z\right)$$

$$\vec{E}_{total} \approx 8.65\,\vec{a}_x + 7.24\,\vec{a}_y - 7.24\,\vec{a}_z \ \frac{V}{m}$$

3. a) Symmetries

translation along x & y $\Rightarrow$ no dependence on x or y

rotation about axis $\parallel$ to z $\Rightarrow$ no E_x or E_y components.

b) $\rho = A(a^2 - z^2)$ for $-a < z < a$

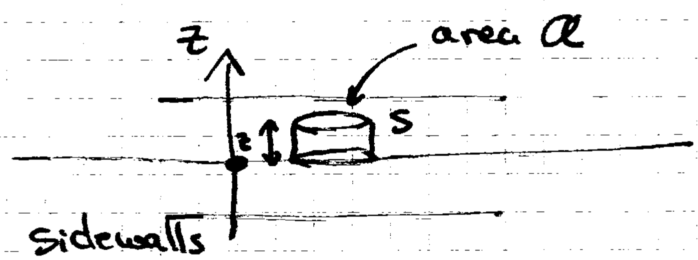
Symmetries $\Rightarrow$ $\vec{E} = E_z(z)\,\vec{a}_z$

also $\vec{E}(z=0) = 0$

Gauss's Law $\epsilon_0 \oint_S \vec{E}\cdot d\vec{s} = Q_{encl}$

Inside the slab

choose S as shown.



notice $\vec{E}\cdot d\vec{s} = 0$ for sidewalls

and $\vec{E}\cdot d\vec{s} = E_z\, ds$ for end caps E_z constant

also $E_z = 0$ for bottom end cap

so $\epsilon_0 \oint_S \vec{E}\cdot d\vec{s} = \epsilon_0\, \mathcal{a}\, E_z$

The charge enclosed depends on the height of S.
Let the height be z

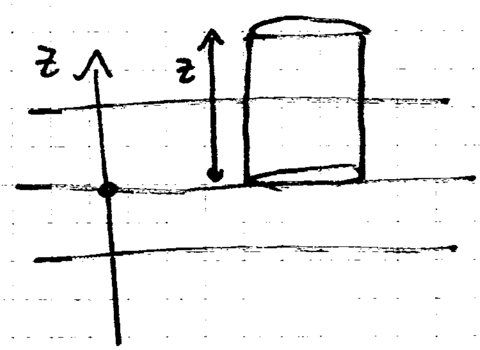
then
$$Q_{encl.} = \iiint_0^z A(a^2 - z'^2)\, dz'\, dx\, dy$$

$$= A\,\mathcal{A}\left(a^2 z' - \tfrac{1}{3} z'^3\right)\Big|_0^z$$

so
$$\epsilon_0 \mathcal{A} E_z = A\,\mathcal{A}\left(a^2 z - \tfrac{1}{3} z^3\right)$$

so
$$\vec{E} = \frac{\pm A}{\epsilon_0}\left(a^2 z - \tfrac{1}{3} z^3\right)\vec{a}_z \qquad \begin{matrix} + & z > 0 \\ - & z < 0 \end{matrix}$$

Outside the slab



choose S as shown

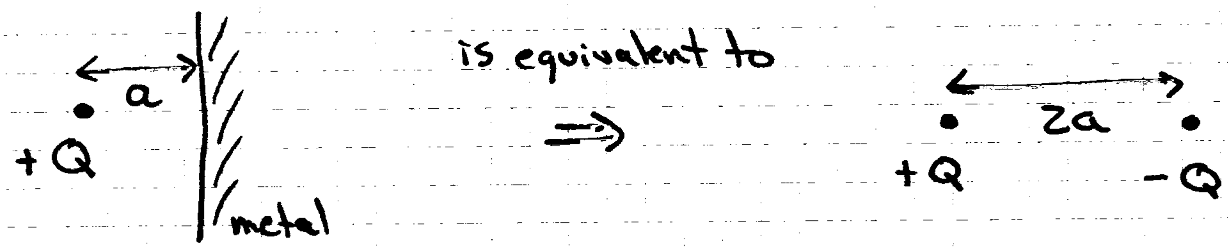
As above
$$\epsilon_0 \oint_S \vec{E}\cdot d\vec{S} = \epsilon_0 \mathcal{A} E_z$$

but now
$$Q_{encl.} = \iiint_0^a A(a^2 - z'^2)\, dz\, dx\, dy$$

$$= A\,\mathcal{A}\left(a^2 z' - \tfrac{1}{3} z'^3\right)\Big|_0^a = A\,\mathcal{A}\,\tfrac{2}{3} a^3$$

so
$$\vec{E} = \frac{\pm A}{\epsilon_0}\,\frac{2}{3} a^3\, \vec{a}_z \qquad \begin{matrix} + & z > 0 \\ - & z < 0 \end{matrix}$$

4. a) Use the method of image charges

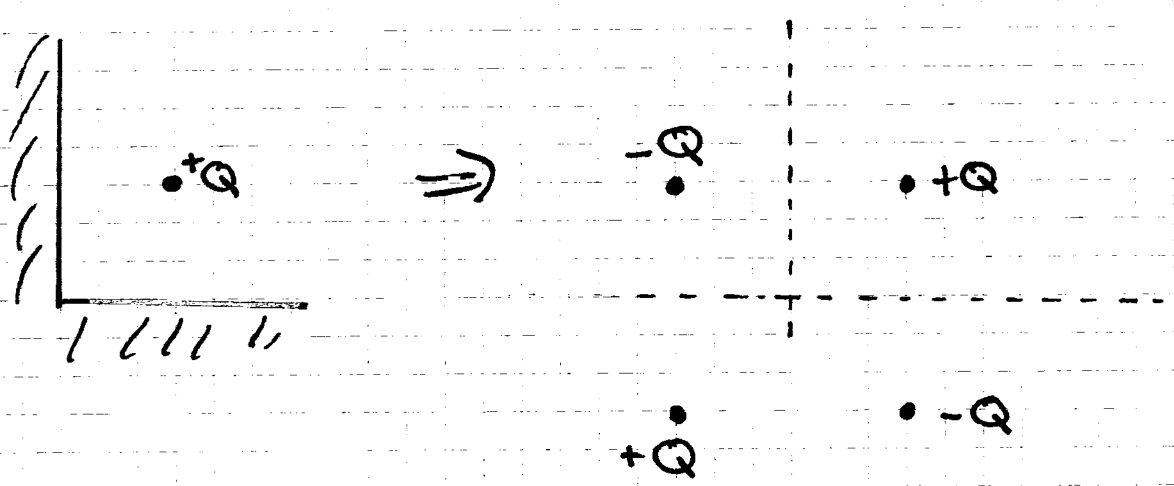


The force on Q is thus $F = \dfrac{-Q^2}{4\pi\epsilon_0 (2a)^2}$

for $Q = 1\,nC$ and $a = 1\,cm$

$F = 2.25 \times 10^{-5}$ Nt attractive to the metal

b) The appropriate image charges are as shown



This arrangement makes both planes of metal equipotentials.

5. The maximum electric field is $E = 1.5\times10^{6}\ \frac{V}{cm}$

$= 1.5\times10^{8}\ V/m$

since $\vec{D} = \epsilon_0 \epsilon_R \vec{E}$

so $D = 4.25\times10^{-3}\ \frac{C}{m^2}$

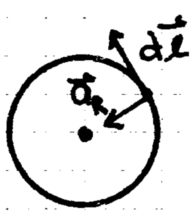
total volume of capacitor

$$V = (25\times10^{-6}\,m)(0.1\,m)^2 = 2.5\times10^{-7}\,m^3$$

total energy stored

$$W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,dV = \frac{1}{2}\left(1.5\times10^{8}\,\frac{V}{m}\right)\left(4.25\times10^{-3}\,\frac{C}{m^2}\right)\cdot\left(2.5\times10^{-7}\,m^3\right)$$

$$W_{max} = 7.97\times10^{-2}\ \text{Joules.}$$

6. 

Biot-Savart $\vec{H} = \frac{1}{4\pi}\oint \frac{I\,d\vec{l}\times\vec{a}_R}{R^2}$

$$d\vec{l} = a\,d\varphi\ \vec{a}_\varphi \qquad \vec{a}_R = -\vec{a}_\rho$$

so $d\vec{l}\times\vec{a}_R = a\,d\varphi\,\vec{a}_z$ also $R=a$ for the loop.

so $$\vec{H} = \frac{I}{4\pi}\int_0^{2\pi}\frac{a\,d\varphi}{a^2}\,\vec{a}_z = \frac{I}{2a}\,\vec{a}_z$$